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ACOUSTIC OSCILLATIONS NEAR A THIN-WALLED CYLINDRICAL OBSTACLE IN A CHANNEL

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The problems of existence of eigenoscillations in infinite cylindrical regions comprising a thin cylindrical obstacle are studied. The existence criteria for eigenoscillations are obtained. For obstacles allowing axial symmetry, the dependence of eigenoscillation frequencies on the obstacle dimensions is studied. The form of eigenoscillations is studied for the first modes.

Introduction. The existence of acoustic eigenoscillations in a certain structure can lead to the occurrence of resonance effects — the substantial growth of the oscillation amplitudes because of the feedback to a nonlinear source. Therefore, a study of acoustic eigenoscillations is of great importance for applications. A cylindrical channel comprising a thin-walled obstacle is widely used in engineering (turbines, fans, and pipe lines). As a rule, the oscillations which occur in real structures are due to a nonlinear source (the formation of coherent structures in a flow of a liquid, vibrational combustion, etc.). The unbounded regions of oscillations are the most frequent. The condition of the onset of intense oscillations is the closeness of the source oscillation frequency to the eigenfrequency of acoustic oscillations in the unbounded region. In addition, the form of acoustic perturbations caused by the source should be matched to the form of the corresponding eigenfunction (the nonorthogonality condition of the driving force and the mode of eigenoscillations), thereby determining the importance of our study.

The main difficulty in describing eigenoscillations in unbounded regions is that the eigenfrequencies of acoustic oscillations are immersed into a continuous spectrum of frequencies corresponding to the generalized eigenoscillations. Sukhinin [1] resolved this difficulty by restricting the class of admissible solutions and proved the existence of eigenoscillations for sufficiently long obstacles. In the present study, eigenoscillations are shown to always exist for obstacles of a certain kind. In addition, numerical calculations of the form of eigenfunctions and the dependences of the eigenfunctions on some linear parameters are performed.

1. Formulation of the Problem. Let (x, y, z) be the Cartesian coordinates of the three-dimensional space $\mathbb{R}^3$. An infinite cylindrical channel is described by a directrix $\Gamma = \{(x,y)\colon G(x,y) = 0\}$ and a generatrix parallel to the Oz axis. The directrix of the cylindrical channel is assumed to be smooth, bounded, and limited at the surface (x, y). The notation $Z_\Gamma = \Gamma \times \mathbb{R}$ is adopted to describe the cylindrical channel. The obstacle in the channel is assumed to be bounded, infinitely thin, and cylindrical and with a generatrix parallel to the Oz axis, and it is described by means of the directrix $\gamma = \{(x,y)\colon g(x,y) = 0\}$ on the plane (x,y) and the system of inequalities $-\infty < a(x,y) \leqslant z \leqslant b(x,y) < +\infty$, $(x,y) \in \gamma$. The latter means that the obstacle edges can be irregular. For the description of the obstacle surface, the notation $Z_\gamma = \{(x,y,z)\colon g(x,y) = 0,\ a(x,y) \leqslant z \leqslant b(x,y)\}$ is adopted. It is assumed that $G(x,y)$, $g(x,y)$, $a(x,y)$, and $b(x,y)$ are sufficiently smooth functions. The directrix of the cylindrical obstacle is assumed to be inside the generatrix of the channel and divides the region $D = \mathrm{Int}(\Gamma)$ limited by Γ into a few coupled parts $D = D_1 \cup D_2 \cup \ldots \cup D_N \cup \Gamma$. The possible locations of the directrices of the cylindrical channel and the obstacle are given in Fig. 1 (the case presented in Fig. 1b is not considered in this paper).

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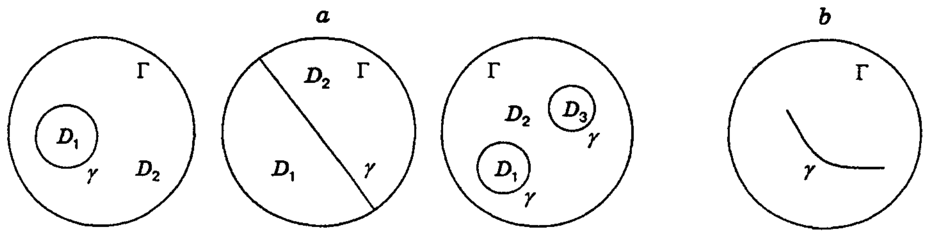

Fig. 1

Equations which Describe Acoustic Oscillations. The potential $u(x,y,z,t)$ of the acoustic velocity perturbation is assumed to be periodically dependent of time: $u(x,y,z,t) = u(x,y,z,t)\exp(i\omega t)$; therefore, one can consider that the equations for the potential of the acoustic velocity perturbation $u(x,y,z,t)$ have the form

$$u_{xx} + u_{yy} + u_{zz} + \lambda^2 u = 0 \quad \text{in} \quad \Omega = D \times \mathbb{R} \backslash Z_\gamma. \tag{1.1}$$

Here the dimensionless frequency λ and the spatial variables x and y have the form $\lambda = \omega H/c$, $x = \tilde{x}/H$, and $y = \tilde{y}/H$, where c is the speed of sound, H is the characteristic transverse size of the cylindrical channel, and ω is the cyclic frequency of acoustic oscillations. In dimensionless variables, the cross-sectional area of the channel is assumed to be equal to unity, and the dimensionless quantity L characterizes the obstacle length with respect to the channel cross section. The following no-slip conditions should be satisfied on the obstacle Z_γ and the channel walls Z_Γ:

$$u_n = 0 \quad \text{on} \quad Z_\Gamma \cup Z_\gamma. \tag{1.2}$$

Here $\boldsymbol{n}$ is the vector of the normal to the corresponding surface. According to the physical meaning of the problem, its solution $u(x,y,z)$ requires the satisfaction of the condition of energy finiteness in the entire region of oscillations

$$E(u) = \int_\Omega [|u|^2 + |\nabla u|^2]\, d\Omega < \infty. \tag{1.3}$$

Here $\Omega = D \times \mathbb{R} \backslash Z_\gamma$ is the region of oscillations; $E(u)$ takes the meaning of the energy of oscillations.

Radiation Conditions and Continuous Spectrum. In the general case, solution (1.1) describes the incoming and outgoing waves, which leads to nonunique solutions in the unbounded regions; therefore, additional conditions based on the physical meaning of the problem are required.

Definition 1.1. Solution (1.1) satisfies the radiation condition if, for some sufficiently small ε [$\varepsilon > 0$, and $-1/\varepsilon < a(x,y) \leqslant x \leqslant b(x,y) < 1/\varepsilon$], the representations

$$u(x,y) = \begin{cases} c_0^{(+)} \exp(i\lambda z) + \sum\limits_{k=1}^{\infty} c_k^{(+)} Y_k(x,y) \exp\left(-z\sqrt{\beta_k^2 - \lambda^2}\right) & (z \geqslant 1/\varepsilon), \\ c_0^{(-)} \exp(-i\lambda z) + \sum\limits_{k=1}^{\infty} c_k^{(-)} Y_k(x,y) \exp\left(z\sqrt{\beta_k^2 - \lambda^2}\right) & (z \leqslant -1/\varepsilon) \end{cases} \tag{1.4}$$

are valid. It is assumed that, for $\lambda^2 < \beta_k^2$, a branch of the quadratic root is chosen such that $\sqrt{\beta_k^2 - \lambda^2} > 0$, $c_k^{(-)}$ and $c_k^{(+)}$ are the complex numbers such that the series (1.4) converges, and $Y_k(x,y)$ and β_k^2 ($k = 0, 1, 2, \ldots$) are the eigenfunctions and eigenvalues of the Laplace operator in the region D with the Neumann conditions on the boundary Γ enumerated in the ascending order of eigenvalues. It is worth mentioning that, if a function satisfies the radiation condition, it either increases or decreases in the general case just as the exponent with distance from the origin of coordinates (the obstacle). The radiation conditions are discussed in more details in [2].

In the class of functions satisfying the radiation conditions (1.4), problem (1.1), (1.2) is a Fredholm function [2] having nontrivial solutions only for a set Λ^* discrete at some Riemann surface Λ [the values of the parameter λ of Eq. (1.1)]. In [2], they are called quasi-eigenvalues of the problem, and the solutions u^* of the problem (1.1), (1.2), which correspond to λ_* ($\lambda_* \in \Lambda^*$), are called quasi-eigenfunctions. In the case of the finite energy of quasi-eigenoscillations [$E(u^*) < \infty$], the quasi-eigenfunction (u^*) describes the classical eigenoscillations located in the vicinity of the obstacle and could cause resonance phenomena. In the case where the energy of quasi-eigenoscillations is infinite, the physical meaning of quasi-eigenoscillations is not yet completely clear.

If $\lambda^2 \geqslant \beta_k^2$, then $Y_k(x,y)\exp(\pm iz\sqrt{\lambda^2-\beta_k^2})$ ($k = 0, 1, \ldots$) describes the generalized eigenwaves in a channel having no obstacle. If $k = 0$, these waves exist for all real values of the dimensionless frequency λ. This means that the self-conjugated expansion of the Laplace operator $-\Delta$, which corresponds to problem (1.1), (1.2), has a continuous spectrum filling the entire nonnegative section of the real straight line. The point spectrum of the operator is submerged into the continuous spectrum and corresponds to the values of λ_*^2, for which there exists a nontrivial solution of problem (1.1)–(1.3). For further consideration, we need

Definition 1.2. The generalized eigenwaves in a channel which are described by the functions $W_0 = \exp(i\lambda z)$ are called piston modes.

Remark 1.1. Since the function that describes the piston mode does not depend on the variables (x,y), the piston mode is a generalized eigenfunction of the channel with and without a thin-walled cylindrical obstacle.

Restriction of the Class of Solutions. The operator $-\Delta_N$, which corresponds to the problem of eigenoscillations near an obstacle in a channel, has a continuous spectrum coinciding with the positive semi-axis of real numbers, thereby complicating a study of the eigenfrequencies and eigenfunctions by variational methods. The restriction of the space of admissible solutions can shift the lower edge ν of the continuous spectrum σ_1 of the operator $-\Delta_N^{(1)}$ from the origin of coordinates. This makes it possible to use variational methods within the interval $(0,\nu)$.

Relations (1.1) and (1.2) are the Neumann problem for the Helmholtz equation, which is called below an EO (eigenoscillations) problem, and H_s is the space of admissible solutions of the problem (Sobolev's space).

Definition 1.3. The solution u^* of the EO problem which satisfies (1.3) is called an eigenfunction of this problem. The corresponding frequency λ^* is called an eigenfrequency.

We note the following: the eigenfunctions of the EO problem are localized in the vicinity of the obstacle, the eigenfrequencies are submerged into the continuous spectrum, the piston mode $W_0 = \exp(i\lambda z)$ is a generalized eigenfunction of the EO problem, and the eigenfrequencies and eigenfunctions of the EO problem describe the acoustic resonance phenomena near the obstacle in the channel. Owing to the results of the theory of self-conjugated operators, the eigenfunctions have a zeroth projection in a certain space of functions onto the piston mode, since it is a generalized eigenfunction. Therefore, if the eigenfunction u^* of the EO problem exists, it should satisfy the necessary condition

$$\int_\Omega \exp(i\lambda z)u^*(x,y,z)\,d\Omega = 0$$

for all the values of λ. This condition is satisfied for all λ if and only if the equality

$$\int_D u^*(x,y,z)\,dx\,dy \equiv 0 \tag{1.5}$$

holds for all z. This condition bounds the space H_s of admissible solutions of the EO problem to the space that is the subspace H_s ($H_0 \subset H_s$). Hereafter, the EO problem with the additional condition (1.5) satisfied for all z is called an EOO (eigenoscillations orthogonal to the piston mode) problem. The continuous spectrum σ_1 corresponding to the EOO problem has the form $\sigma_1 = [\beta_1^2, \infty)$; therefore, the eigenvalues are searched for within the interval $(0, \beta_1^2)$.

2. Existence and Form of Eigenfunctions. The form of eigenfunctions far from the obstacle is described with the use of the radiation (the absence of outgoing waves) and energy-finiteness conditions. To gain an insight into the mechanics of eigenoscillations and the development of numerical algorithms, one has to know the form of the eigenfunction in the vicinity of the obstacle.

Form of the Eigenfunction in the Vicinity of the Obstacle and Its Edges. The physical assumptions for a study of the form of eigenfunctions in the vicinity of the edge are as follows: (a) the energy in the vicinity of the edge is finite; (b) the edge does not radiate.

Remark 2.1. The assumptions (a) and (b) are equivalent and are the consequence of (1.3).

Let L be such that, for all $(x,y) \in \gamma$, the inequalities $a(x,y) < -L/2$, and $L/2 < b(x,y)$ hold. The quantity L can be considered as a characteristic dimensionless length of the obstacle Z_γ. Let (ρ,φ,z) be the cylindrical coordinates in the vicinity of a fixed point belonging to the obstacle edge, where ρ is the distance from the point of the obstacle edge, in the vicinity of which the form of solution is studied, to the current point in the plane (ρ,φ), and φ be the angle measured from the inner surface of the obstacle. One checks by direct expansion over the small parameter ρ that the solution u^* of the EO (or EOO) problem has the form

$$u^* = \text{const}\,\sqrt{\rho}\cos(\varphi/2) \tag{2.1}$$

in the vicinity of the obstacle edge. In terms of Ω, the solution u of the EO (or EOO) problem can be given in the form

$$u = u_{\rm d} + u_{\rm c} \tag{2.2}$$

Here $u_{\rm d}$ is a discontinuous function on the obstacle Z_γ and $u_{\rm c}$ is a continuous function in $\Omega \cup Z_\gamma$ (inside the cylindrical channel). If $a(x,y) = \text{const}$ and $b(x,y) = \text{const}$, the EOO problem becomes itself (invariant) with respect to the change of the variables $z \to -z$ with an appropriate choice of the origin of coordinates ($x = 0$ is the middle of the obstacle). Therefore, any solution u of this problem can be represented in the form $u = u_s + u_a$, where $u_s(x,y,z) = u_s(x,y,-z)$ and $u_a(x,y,z) = -u_a(x,y,-z)$ are the symmetric (even) and antisymmetric (odd) components of u with respect to z. Since the EOO problem is linear, the space H_0 of all admissible solutions of the problem can be represented in the form of a direct sum of two spaces of the functions $H_0 = H_\sigma \oplus H_\alpha$ which are symmetric H_σ and antisymmetric H_α with respect to z. Owing to the linearity, the EOO problem is divided into two independent problems for even and odd functions with respect to z. Further consideration concerns solutions of the EOO problem that are symmetric (even) with respect to z, unless otherwise specified.

Existence of Eigenoscillations. To substantiate the correctness of the mathematical description of acoustic eigenoscillations near a thin-walled cylindrical obstacle in a channel, we must show that the eigenoscillations exist at least for some geometric parameters in the cylindrical channel and are described by the mathematical model suggested. To this end, the "Dirichlet–Neumann bracket" method [3] is used Let, in addition to the conditions of the EOO problem, the Dirichlet (D) $u(x,y,z) = 0$ or the Neumann (N) condition $\partial u/\partial z = 0$ be satisfied at the additional boundaries $\Xi = \{(x,y,z): |z| = 1/\varepsilon, (x,y,z) \in \Omega\}$. For convenience, the EOO problem with condition (D) is denoted by EOO(Dε), and with condition (N) by EOO(Nε). Let $\lambda_{\mathrm{D}\varepsilon}$, $u_{\mathrm{D}\varepsilon}$ and $\lambda_{\mathrm{N}\varepsilon}$, $u_{\mathrm{N}\varepsilon}$ be the eigenvalues and eigenfunctions of the EOO(Dε) and EOO(Nε) problems, respectively. Since the condition (N) expands the space of admissible solutions of the EOO problem and the condition (D) restricts the space, the inequalities that can be obtained by means of variational formulations of the problems [3]

$$\lambda_{\mathrm{N}\varepsilon} \leqslant \lambda^* \leqslant \lambda_{\mathrm{D}\varepsilon} \tag{2.3}$$

are valid for all ε $(L/2 < 1/\varepsilon)$.

Remark 2.2. If the inequalities $0 < \lambda_{\mathrm{N}\varepsilon}$ and $0 < \lambda_{\mathrm{D}\varepsilon} < \beta_1$ are satisfied for ε $(1/\varepsilon > L/2)$, the existence of the eigenvalue of the EO problem follows from (2.3). If $1/\varepsilon = L/2$, the condition (D) is a "soft" radiation condition for oscillations in the channel and the circular channel (regions I and II, respectively, in Fig. 2). In this case, $\lambda_{\mathrm{D}\varepsilon} = \pi/L$. Therefore, if $\pi/L < \beta_1$, we have $\lambda^* < \beta_1$. The latter inequality will be satisfied, if the dimensionless length of the obstacle L satisfies the inequality $L > \pi/\beta_1$. Since the orthogonality conditions

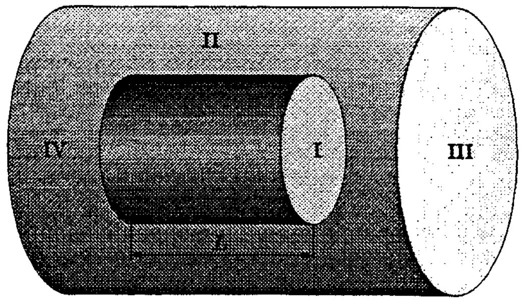

Fig. 2

of the admissible solutions of the EOO problem with respect to the piston mode (1.5) are satisfied, $0 < \lambda_{N\varepsilon}$ holds for $\varepsilon < 2/L$. This means that, for the Neumann problem in the domain $\Omega \cap \{(x, y, z)\colon |z| < 1/\varepsilon\}$, in the space of the admissible solutions of the EOO problem, the first eigenvalue of the Laplace operator is rigorously greater than zero. As a result, the following theorem is valid [1].

Theorem 2.1 (the sufficient condition for the existence of eigenoscillations near an obstacle in a channel). *If the dimensionless length of the obstacle is $L > \pi/\beta_1$, there exist nontrivial eigenfrequencies of the EOO problem.*

This theorem answers the question of the existence of eigenfrequencies of the problem for sufficiently large relative lengths of the obstacle. A more general statement is also true.

Theorem 2.2 (existence of eigenoscillations). *If $a(x, y) \leqslant a_0 < b_0 \leqslant b(x, y)$, eigenoscillations exist near an obstacle Z_γ in a channel Z_Γ.*

Proof. The origin of coordinates can be chosen in such a way that $a(x, y) \leqslant -L/2 < 0 < L/2 \leqslant b(x, y)$. It suffices to show that the inequalities

$$0 < \lambda_{N\varepsilon} \leqslant \lambda^* \leqslant \lambda_{D\varepsilon} < \beta_1 \tag{2.4}$$

are satisfied for some $\varepsilon > 0$.

1. *Estimate from Below.* If $1/\varepsilon > L/2$, then $0 < \lambda_{N\varepsilon}$ by virtue of (1.5) and the connectedness $\Omega \cap \{(x, y, z)\colon |z| < 1/\varepsilon\}$.

2. *Estimate from Above.* Let the continuous component u_c of the approximate eigenfunction u in (2.2) have the form $u_c = \cos(\varepsilon\pi z/2) Y_1(x, y)$ in Ω.

The component of the approximate discontinuous eigenfunction (2.2) on the obstacle has the form

$$u_d = \begin{cases} æ(S-1)(z - L/2), & (x, y) \in D_1, \\ æ(z - L/2), & (x, y) \in D_2, \end{cases}$$

where æ is an arbitrary constant and S is the area D_1; the area D is assumed to be equal to unity. The function u_d can be considered defined in the entire region of oscillations if it is taken to be zero beyond regions I and II. The inequality

$$(\lambda_{D\varepsilon})^2 \leqslant \int\limits_{\Omega_\varepsilon} |\nabla(u_c + u_d)|^2 \, d\Omega_\varepsilon \Big/ \int\limits_{\Omega_\varepsilon} |u_c + u_d|^2 \, d\Omega_\varepsilon = \mu^2(æ, \varepsilon), \tag{2.5}$$

which reflects the variational property of eigenvalues [3], is valid for all æ. Here $\Omega_\varepsilon = \Omega \cap \{(x, y, z)\colon |z| \leqslant 1/\varepsilon\}$. By direct calculation (as a consequence of the finite carrier function u_d), for small ε, one checks that the asymptotic representation

$$\mu^2(æ, \varepsilon) \cong \beta_1^2 + A\varepsilon + B\varepsilon^2 \tag{2.6}$$

holds. The quantities A and B depend on æ. Since ε and æ are independent, for small ε, A is the determining quantity in expansion (2.6). It is true that

$$A = LS\text{æ}^2(12 - L^2\beta_1^2)(1-S)\Big/\Big(12\iint\limits_D Y_1^2(x,y)\,dx\,dy\Big)$$

$$+\text{æ}L^2\beta_1^2\iint\limits_D Y_1^2(x,y)\,dx\,dy\Big/\Big(2\iint\limits_D Y_1^2(x,y)\,dx\,dy\Big).$$

Hence, we have $A < 0$ for small æ, positive or negative. Therefore, for sufficiently small ε and æ, we have that $\mu^2(\text{æ}, \varepsilon) < \beta_1^2$. By virtue of relation (2.5), inequalities (2.4) hold. Theorem 2.1 is proved.

Remark 2.3. The method of proving Theorem 2.2 is based on the evaluation of the perturbation of the generalized eigenfunction introduced by the obstacle. The similarity of the eigenfunction to the generalized eigenfunction $Y_1(x,y)\exp(iz\sqrt{\lambda^2-\beta_1^2})$ increases with decrease in the obstacle length.

Remark 2.4. In principle, the mechanics of eigenoscillations near long ($L > \pi/\beta_1$) and short ($L \ll \pi/\beta_1$) obstacles Z_γ is not different. For $L \ll \pi/\beta_1$, the eigenfunctions cannot be localized between the obstacle and channel walls or inside the obstacle. If $S \approx 0$ (or $S \approx 1$), the eigenfunction is "forced" out of the space between the obstacle and the wall and from the obstacle "inside" (the smallness of æ) as $L \to 0$. If $L > \pi/\beta_1$, as follows from the proof of Theorem 2.1, the eigenfunction is localized inside the obstacle and between the obstacle and the channel wall.

Here we discuss the behavior of the smallest eigenvalues except for the cases specified.

3. Radial Oscillations. Eigenoscillations near a Thin-Walled Round Cylindrical Obstacle in a Circular Channel. A thin-walled round cylindrical obstacle in a circular channel is typical of engineering problems. Let (r,φ,z) be the cylindrical system of coordinates whose axis passes through the centers of the obstacle and channel directrices, which are the concentric circles in the plane $z = \text{const}$. The directrices Z_Γ and Z_γ have the forms $\Gamma = \{(r,\varphi,z): r = 1, z = \text{const}\}$ and $\gamma = \{(r,\varphi,z): r = h, z = \text{const}\}$. The edges Z_γ are assumed to be smooth and located in the planes $z = -L/2$ and $z = L/2$, and the origin of coordinates is in the middle of the channel.

Representation of Eigenfunctions. Since the EO and EOO problems have axial symmetry with the axis of rotation z, the solutions of these problems can be assumed to be independent of the angular coordinate φ and are only functions of (r,z). It is convenient to divide the oscillation region Ω into the following subregions: I) $\{(r,\varphi,z): r < h, -L/2 < z < L/2\}$; II) $\{(r,\varphi,z): h < r < 1, -L/2 < z < L/2\}$; III) $\{(r,\varphi,z): r < 1, z < -L/2\}$; IV) $\{(r,\varphi,z): r < 1, L/2 < z\}$ (Fig. 2). Let u_j $(j = 1,\dots,4)$ be the restriction of the solution u of the EO (or EOO) problem in I–IV, respectively. The general solution of the EO problem in I–IV for even and odd functions with respect to the variable z has the form

$$u_1(r,z) = a_0\left\{\begin{array}{c}\cos(\lambda z)\\ \sin(\lambda z)\end{array}\right\} + \sum_{m=1}^{\infty} a_m J_0\Big(\frac{r\beta_m}{h}\Big)\left\{\begin{array}{c}\cosh\big(z\sqrt{\beta_m^2/h^2-\lambda^2}\big)\\ \sinh\big(z\sqrt{\beta_m^2/h^2-\lambda^2}\big)\end{array}\right\},$$

$$u_2(r,z) = b_0\left\{\begin{array}{c}\cos(\lambda z)\\ \sin(\lambda z)\end{array}\right\} + \sum_{m=1}^{\infty} b_m R_m(r)\left\{\begin{array}{c}\cosh\big(z\sqrt{\sigma_m^2-\lambda^2}\big)\\ \sinh\big(z\sqrt{\sigma_m^2-\lambda^2}\big)\end{array}\right\}, \tag{3.1}$$

$$u_3(r,z) = c_0\exp(i\lambda z) + \sum_{k=1}^{\infty} c_k J_0(r\beta_k)\exp\big(-z\sqrt{\beta_k^2-\lambda^2}\big), \qquad u_4(r,z) = \left\{\begin{array}{c}+u_3(r,-z)\\ -u_3(r,-z)\end{array}\right\}.$$

The function $u_4(r,z)$ is expressed in terms of $u_3(r,z)$ with the use of the symmetry and antisymmetry conditions for even and odd oscillation modes with respect to z. Here and below, β_k^2, and σ_k^2 $(k = 1, 2, \dots)$ are the eigenvalues of the Laplace operator in the regions D and D_2 (ring) with Neumann boundary conditions, β_k $(k = 0, 1, 2, \dots)$ are the zeros of the function $\beta J_1(\beta)$, and σ_k^2 $(k = 1, 2, \dots)$ are the zeros of the function $J_1(\sigma)Y_1(\sigma h) - J_1(\sigma h)Y_1(\sigma)$ reckoned in ascending order. The functions $R_m(r)$ $(m = 1, 2, \dots)$ are eigenfunctions of the Neumann problem for the Laplace operator in the region D_2 (ring) and have the form

$R_m(r) = c_m^{(1)} J_0(\sigma_m r) + c_m^{(2)} Y_0(\sigma_m r)$; $c_m^{(1)}$, and $c_m^{(2)}$ are found numerically from the Neumann conditions at the boundaries D_2 for all σ_m. Conditions (1.5) are satisfied if

$$c_0 = 0, \qquad a_0 h^2 + b_0(1 - h^2) = 0. \tag{3.2}$$

For a function of the form (3.1) with conditions (3.2) to be the solution of the EOO problem, at the boundaries of regions I–III, I–IV, II–III and II–IV there should be satisfied the continuity conditions of the solution and its normal derivative, which are usually called matching conditions [4]. By virtue of symmetry of the problem with respect to z, it is enough to satisfy the matching conditions at the boundaries of regions I–III and II–III. Let G_{13} be the boundary between I and III, and G_{23} be the boundary between II and III. These are the sections of straight lines in the space of variables (r, z). The matching conditions have the form

$$u_1 = u_3, \quad \frac{\partial u_1}{\partial z} = \frac{\partial u_3}{\partial z} \quad \text{on} \quad G_{13}, \qquad u_2 = u_3, \quad \frac{\partial u_2}{\partial z} = \frac{\partial u_3}{\partial z} \quad \text{on} \quad G_{23}. \tag{3.3}$$

These conditions mean that the function of the form (3.1) is a weak solution of EO or EOO problems in Sobolev's space. For elliptic equations, the weak solution is a strong solution.

Discretization of the Problem. One can approximate (for symmetric and antisymmetric functions with respect to z) the eigenfunctions $\hat{u}(r, z)$ of the EOO problem in regions I and II (vicinities of the obstacle) in the form

$$\hat{u}_1(r,z) = a_0 \left\{ \begin{array}{c} \cos(\lambda z) \\ \sin(\lambda z) \end{array} \right\} + \sum_{m=1}^{M} a_m J_0\left(\frac{r\beta_m}{h}\right) \left\{ \begin{array}{c} \cosh(z\sqrt{\beta_m^2/h^2 - \lambda^2}) \\ \sinh(z\sqrt{\beta_m^2/h^2 - \lambda^2}) \end{array} \right\},$$

$$\hat{u}_2(r,z) = b_0 \left\{ \begin{array}{c} \cos(\lambda z) \\ \sin(\lambda z) \end{array} \right\} + \sum_{m=1}^{M} b_m R_m(r) \left\{ \begin{array}{c} \cosh(z\sqrt{\sigma_m^2 - \lambda^2}) \\ \sinh(z\sqrt{\sigma_m^2 - \lambda^2}) \end{array} \right\}. \tag{3.4}$$

This representation of the eigenfunctions has $(2M+2)$ unknowns $\{a_m, b_m\}_{m=0,1,\ldots,M}$. The additional relation (3.2) decreases the number of variables by one. If (3.3) is regarded as the equalities of the Fourier series for functions of the form (3.4) in an appropriate orthogonal basis within the interval $G_{13} \cup C_{23}$, they take the form of an infinite homogeneous system of equations [4]. Using the orthogonality in the region D of the Bessel functions $J_0(\beta_m r)$ and $J_0(\beta_k r)$ (for $k \neq m$) and eliminating c_k, one can write relations (3.3) in discretized form

$$\int_0^h \left(\alpha_k \hat{u}_1 + \frac{\partial \hat{u}_1}{\partial z}\right) J_0(\beta_k r) r\, dr + \int_h^1 \left(\alpha_k \hat{u}_2 + \frac{\partial \hat{u}_2}{\partial z}\right) J_0(\beta_k r) r\, dr = 0 \quad (z = L/2,\ k = 1, 2, \ldots), \tag{3.5}$$

where $\alpha_k = \sqrt{\beta_k^2 - \lambda^2}$. The discretization of the problem should take into account approximately all the properties. Taking into account the energy-finiteness conditions is a specific problem characteristic of problems with sharp edges. For the numerical solution (3.5) to take into account the energy-finiteness conditions (1, 3), some additional conditions [4] are required. Here, for the sake of correct calculations, the discretized relations (3.3) are supplemented by the forced energy-fitness condition, which allows a substantial improvement in the accuracy and rate of calculations. By virtue of (2.1), at the edges $z = -L/2$ and $z = L/2$ of the obstacle Z_γ, for even and odd modes of oscillations, the equalities

$$a_0 \left\{ \begin{array}{c} \cos(\pm\lambda L/2) \\ \sin(\pm\lambda L/2) \end{array} \right\} + \sum_{m=1}^{M} a_m J_0(\beta_m) \left\{ \begin{array}{c} \cosh(\pm L/2\sqrt{\beta_m^2/h^2 - \lambda^2}) \\ \sinh(\pm L/2\sqrt{\beta_m^2/h^2 - \lambda^2}) \end{array} \right\} = 0,$$

$$b_0 \left\{ \begin{array}{c} \cos(\pm\lambda L/2) \\ \sin(\pm\lambda L/2) \end{array} \right\} + \sum_{m=1}^{M} b_m R_m(h) \left\{ \begin{array}{c} \cosh(\pm L/2\sqrt{\sigma_m^2 - \lambda^2}) \\ \sinh(\pm L/2\sqrt{\sigma_m^2 - \lambda^2}) \end{array} \right\} = 0 \tag{3.6}$$

are valid.

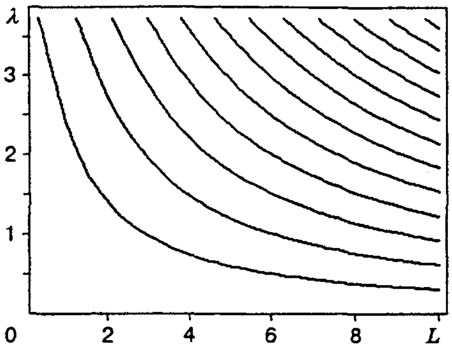

Fig. 3

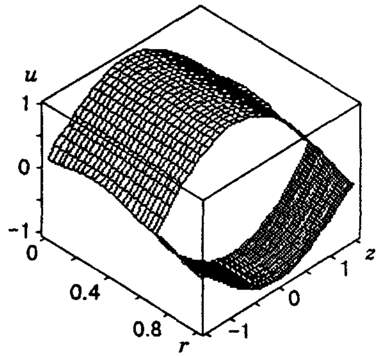

Fig. 4

Relations (3.6) mean the forced satisfaction of the energy-finiteness condition at the obstacle edges for approximate symmetric and antisymmetric eigenfunctions with respect to the z axis and are the supplementary relations for calculations of the eigenvalues and eigenfunctions. Relations (3.2), (3.6), and a certain part of (3.5) form a homogeneous system of $(2M + 2)$ equations for the unknown $\{a_m, b_m\}_{m=0,1,\ldots,M}$. This system describes the approximate eigenfunctions of the EOO problem (3.4). The approximate eigenfunctions for the EOO problem are found from the equality of the determinant of this system to zero.

Numerical Studies. The form of eigenfunctions in the vicinity of the obstacle was studied numerically by means of relations (3.2), (3.5), and (3.6). The studies were performed in the regions $\{(r,z): 0 < r < h, -L/2 < z < L/2\}$ inside and $\{(r,z): h < r < 1, -L/2 < z < L/2\}$ outside the obstacle in the space between the obstacle and the cylindrical channel. By virtue of (1.5), the oscillations inside the obstacle and in the space between the obstacle and the channel walls are in counterphase. This means that (e.g., for the first mode), if the compression phase occurs inside the obstacle, the rarefaction phase occurs between the obstacle and channel walls.

Dependence of the Eigenfrequencies on the Obstacle Length. For a fixed obstacle radius $h = 1/\sqrt{2}$, the dependence of the eigenfrequencies on the obstacle length was studied numerically. A comparison of the numerical results given in Fig. 3 shows that the resonance frequencies decrease as c_N/L with increase in the obstacle length L (c_N is a certain number corresponding to the Nth mode). We note that $\beta_1 = 3.831$ is the threshold frequency for all the oscillation modes (for example, the first eigenfrequency ω_1 tends to 3.831 as $L \to 0$). Except for the first mode, the intervals are determined for the critical length of the obstacle $0 < L < L_*^{(k)}$ ($k = 2, 3, \ldots$) on which the kth mode does not exist. Here $L_*^{(k)} < \infty$ is the maximum (critical) value of the obstacle length (it can be found numerically) for which the kth mode of eigenoscillations does not exist.

The Form of Eigenfunctions. The Amplitude versus the Coordinates. The dependence of the eigenfunction on the coordinates for $L = 3$, and $h = 1/\sqrt{2}$ were calculated by the method of the forced account of the energy finiteness. The dependence of the energy potential on the coordinates for the first (even) mode of eigenoscillations is shown in Fig. 4. Since the velocity potential for steady acoustic oscillations corresponds to an acoustic pressure field, one may consider that Fig. 4 shows the acoustic pressure field of eigenoscillations. Owing to the orthogonality of eigenoscillations to the piston mode, the eigenfunction is antisymmetric with respect to the obstacle if the obstacle is in the middle of the channel, i.e., the oscillations outside and inside the obstacle are in counterphase.

Direction of the Acoustic Flow Velocities. Figure 5 shows the field of acoustic velocities near the obstacle (dashed area) for the first mode of eigenoscillations in the compression phase inside the obstacle and in the rarefaction phase between the obstacle and the cylindrical channel.

Figure 6 shows the pressure field for acoustic eigenoscillations of the first mode in the rarefaction phase inside the obstacle and in the compression phase in the region between the obstacle and the cylindrical channel. We note that the most intense oscillations of the first mode are observed in the middle of the obstacle.

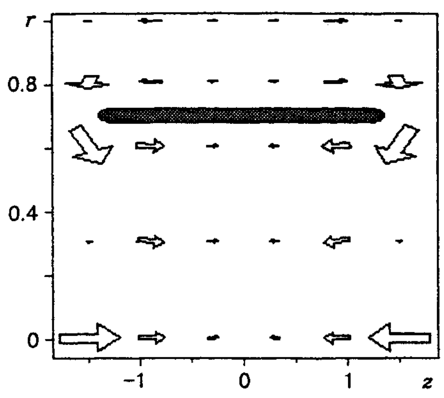

Fig. 5

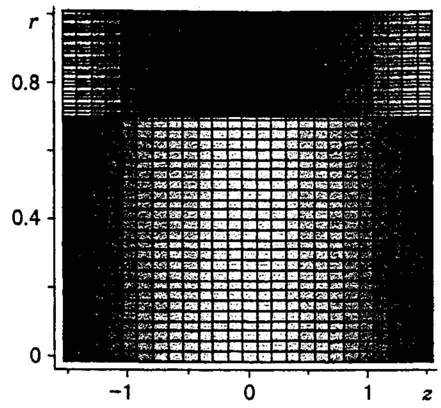

Fig. 6

The studies allow one to better understand the mechanics of eigenoscillations near the obstacle in the channel. From Figs. 5, and 6, it follows that the eigenoscillations are the run-over of a gas from region I to region II and vice versa according to the phase of oscillation.

Conclusions. (1) A mathematical model which describes eigenoscillations near a thin-walled cylindrical obstacle in a channel has been constructed and substantiated. Numerical studies of eigenoscillations are performed.

(2) It has been shown that eigenoscillations exist for any length of an arbitrary obstacle in an arbitrary channel.

(3) The dependence of the frequency of eigenoscillations on the coordinates, the velocity field, the density distribution for the first two modes of eigenoscillations, and the mechanics of oscillations have been calculated numerically.

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